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Note

When a digraph and its line digraph are connected and cospectral [1]

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Abstract

In this paper we characterize all digraphs each one of which is cospectral with its line digraph and both the digraph and its line digraph are connected. Some related enumeration problems are also considered. From these results we can see that there are arbitrarily large sets of cospectral digraphs. © 1998 Elsevier Science B.V. All rights reserved

In 1960, Harary and Norman [3] introduced the concept of line digraph. For a digraph D (loops and multiple arcs are allowed), the *line digraph* $L(D)$ has as its vertex-set the set of arcs of D; (a,b) is an arc of $L(D)$ if and only if there are vertices u,v,w, in D with $a=(u,v)$ and $b=(v,w)$. It is easy to see that a line digraph $L(D)$ has no multiple arcs and it has a loop at a vertex a if and only if a is a loop in D. A natural problem is to determine when D is isomorphic to $L(D)$. In [3], Harary and Norman showed that a connected digraph D is isomorphic to $L(D)$ if and only if every vertex of D has out-degree 1 (say D is functional) or its converse, namely every vertex of D has in-degree 1. When D is an infinite digraph, this problem has been solved by Beineke and Hemminger [2]. This paper attempts to solve a similar problem where a finite connected digraph D (a digraph D with connected underlying graph) is cospectral with its line digraph $L(D)$, and where $L(D)$ is also connected. Furthermore we also consider the related enumeration problem. Finally our results provide a method to produce not only a pair of cospectral digraphs but also a set of cospectral digraphs with arbitrary large cardinality.

Our result is based on the following Lemma obtained by the present authors which solves a problem due to Schwenk and Wilson in [7].

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Lemma 1 (Lin and Zhang [6]). *Let D be a digraph with characteristic polynomial $\chi_D(\lambda)$, then the characteristic polynomial of $L(D)$ is*

$$\chi_{L(D)}(\lambda) = \lambda^{m-n}\chi_D(\lambda),$$

where m is the number of arcs of D and n the number of vertices of D.

Recall that an *out-tree* is a digraph T in which there is a vertex called the root of T being able to reach any other vertex of T by a directed path and the underlying graph of T is a tree. An *in-tree* is the converse of an out-tree. Note that for the degenerate case T has only one vertex which can be considered as both in-tree and out-tree.

The following concept plays an important role in the study of line digraph iteration [5]. D is called an *eddy digraph* if it consists of a directed circuit $C = v_0v_1 \cdots v_{k-1}v_0$ together with an out-tree A_i and an in-tree B_i rooted at each vertex v_i, $0 \leqslant i \leqslant k-1$. Note that in the degenerate case, when $k = 1$, D has a loop. Obviously if a digraph is functional then it must be an eddy digraph. If C is not a directed cycle but its underlying graph is a cycle, we call it a *weak cycle*. A weak cycle can attach some rooted trees at each vertex in the previous way. We call such a digraph a *near eddy digraph*. The other terminologies follow [5].

A vertex of a digraph is called a *sink* (*source*) if its out-degree (in-degree) is 0. The following lemma is obvious.

Lemma 2. *Let D be a digraph with sink* (*source*) *v, then its characteristic polynomial*

$$\chi_D(\lambda) = \lambda\chi_{D-v}(\lambda).$$

A vertex v is an *end vertex* if its in-degree (out-degree) is 1 (0) or out-degree (in-degree) is 1 (0).

Now we are in the position to give the main result.

Theorem 3. *Let D be a digraph with n vertices. Then D and $L(D)$ have the same characteristic polynomial and both D and $L(D)$ are connected if and only if*

(a) *D is an eddy digraph, or*

(b) *D is a near eddy digraph and there is at most one sink or source in the vertices of the weak cycle of D and there is no nondegenerate directed tree attached to the sink or source.*

Proof. By Lemma 1 if D and $L(D)$ have the same characteristic polynomial, then D has the same number of vertices and arcs. Since D is connected the underlying graph of D is unicyclic. Hence D has exact one directed cycle or weak cycle.

(a) If D has a directed cycle C, the other vertices of D are reached from C by a directed path or reach C by a directed path, namely there is no sink or source which is not an end vertex. In fact, if otherwise, $L(D)$ must be disconnected. Hence D is an eddy digraph.

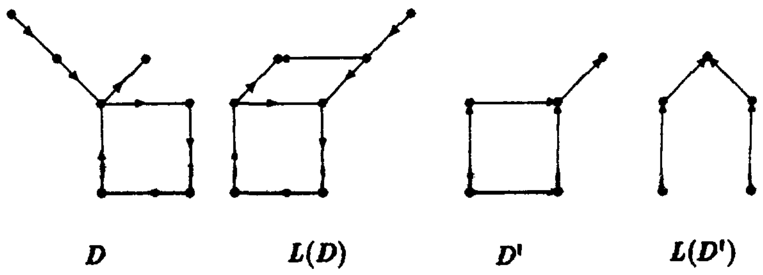

Fig. 1.

(b) If D has a weak cycle, similar to case (a), D is a near eddy digraph. Furthermore, if D has more than one source or sink in the vertices of C or if D has one source (sink) in the vertices of C and there is an out(in)-tree attached to the source (sink) then, $L(D)$ must be disconnected. A contradiction.

Conversely if D is an eddy digraph or near eddy digraph satisfying the condition, it is not difficult to see that both D and $L(D)$ are connected and they have the same characteristic polynomial. □

Two simple examples are given in Fig. 1.

The following theorem determines the spectral of two types of the graphs in Theorem 3.

Theorem 4. *Let D be an eddy digraph with n vertices. Then the eigenvalues of D are*

$$\exp 2j\pi i/k, \quad j=0,1,\ldots,k-1$$

and 0 (*with multiplicity* $n-k$), *where k is the size of the directed cycle of D.*

And if D is a near eddy digraph, then the only eigenvalue of D is 0 (*with multiplicity* n).

Proof. For an eddy digraph, using Lemma 2 successfully we get that the characteristic polynomial is

$$\chi_D(\lambda)=\lambda^{n-k}\chi_{C_k}(\lambda).$$

Since the eigenvalues of C_k are

$$\exp 2j\pi i/k, \quad j=0,1,\ldots,k-1.$$

Our first conclusion follows. The second conclusion can be obtained similarly by Lemma 2. □

Some information about the line digraph iteration can be obtained from the previous theorems. In case (a) D is an eddy digraph, hence D is periodic. For case (b), D is a near eddy digraph, hence L^kD is null. For the details see [5].

Now we turn to the enumeration problem. Since there is a bijection between the rooted trees and in-trees (out-trees), by a well-known result on the counting series of rooted trees (see [4]), we have the counting series for in-trees (out-trees) as follows:

$$T(x)=\sum_{i=1}^{\infty} T_i x^i = x\exp\left(\sum_{k=1}^{\infty} T(x^k)/k\right),$$

where T_i is the number of in-trees (out-trees) of order i. Furthermore the T_i satisfy the following recurrence relation

$$T_{i+1}=i^{-1}\sum_{k=1}^{i} ka_k T_{i-k+1},$$

where

$$a_j=j^{-1}\sum_{d|j} dT_d.$$

for $i\leqslant 26$, the value of T_i can be found in appendix of [4]. We give a few terms here.

$$\begin{aligned} T(x) &= x+x^2+2x^3+4x^4+9x^5+20x^6+48x^7 \\ &\quad +115x^8+286x^9+719x^{10}+\cdots. \end{aligned} \tag{1}$$

Now we give the following

Theorem 5. *The counting series $A_n(x)$ of eddy digraphs whose directed circuit has length n $(n>1)$ is*

$$A_n(x)=Z(C_n,\tilde{T}(x))=\frac{1}{n}\sum_{k|n}\varphi(k)(\tilde{T}(x^k))^{n/k}, \tag{2}$$

where $\varphi(k)$ is the Euler φ function and

$$\tilde{T}(x)=\frac{1}{x}T^2(x)=x+2x^2+5x^3+12x^4+30x^5+74x^6+188x^7+\cdots. \tag{3}$$

Proof. Let us identify the root of an in-tree and out-tree (each one may degenerate to a vertex). The resulting digraph has the counting series

$$\tilde{T}(x)=\frac{1}{x}T^2(x).$$

Substituting (1) into this expression we obtain (3). Since the directed circuit with length n has automorphic group C_n (cyclic group of degree n), its cycle index is

$$Z(C_n)=n^{-1}\sum_{k|n}\varphi(k)S_k^{n/k}.$$

By Polya's theorem (see [4]), (2) is proved. □

Example. Letting $n=4$, we have

$$A_4(x)=\frac{1}{4}(\tilde{T}^4(x)+\tilde{T}^2(x^2)+2\tilde{T}(x^4)).$$

Usually two digraphs are called cospectral if they have the same characteristic polynomial but not isomorphic. The following corollary gives the counting series of digraphs of type (a) producing cospectral digraphs.

Corollary 6. *The counting series $P_n(x)$ of eddy digraphs each of which is cospectral with its own lin: digraph and has a directed circuit with length n is*

$$P_n(x)=\frac{1}{n}\sum_{k|n}\varphi(k)((\tilde{T}(x^k))^{\frac{n}{k}}-2(T(x^k))^{\frac{n}{k}}). \tag{4}$$

Proof. If D is a functional graph, then D consists of a directed circuit C together with an in-tree rooted at each vertex of C. As in the proof of Theorem 5, we can see that its counting series is (see [4])

$$Q_n(x)=\frac{1}{n}\sum_{k|n}\varphi(k)T(x^k)^{n/k}. \tag{5}$$

Between functional graphs and their converses, we have

$$P_n(x)=A_n(x)-2Q_n(x), \quad n>1. \tag{6}$$

Substituting (2) and (5) into (6), we obtain (4). □

The enumeration problem of near eddy digraphs is not so simple since there various possibilities for the automorphism group of a weak cycle (see Fig. 2). Note that each such automorphism group must be a subgraph of the dihedral group D_{nh}.

For some weak cycles with n vertices, the enumeration problems are equivalent to the so-called colored necklace problem (see [1], p. 230). The result is well-known. For each such a weak cycle C, we attempt to consider how many near eddy digraphs can be produced. Similar to Theorem 5 we have the following theorem.

Theorem 7. *Let C be a weak cycle with automorphism group G. The counting series $A(x)$ of near eddy digraphs produced by C is*

$$A(x)=Z(G,\tilde{T}(x)),$$

where

$$Z(G,f(x))=Z(G,f(x),f(x^2),\dots,f(x^m))$$

and the polynomial $Z(G,y_1,y_2,\dots,y_m)$ is the cycle index of G.

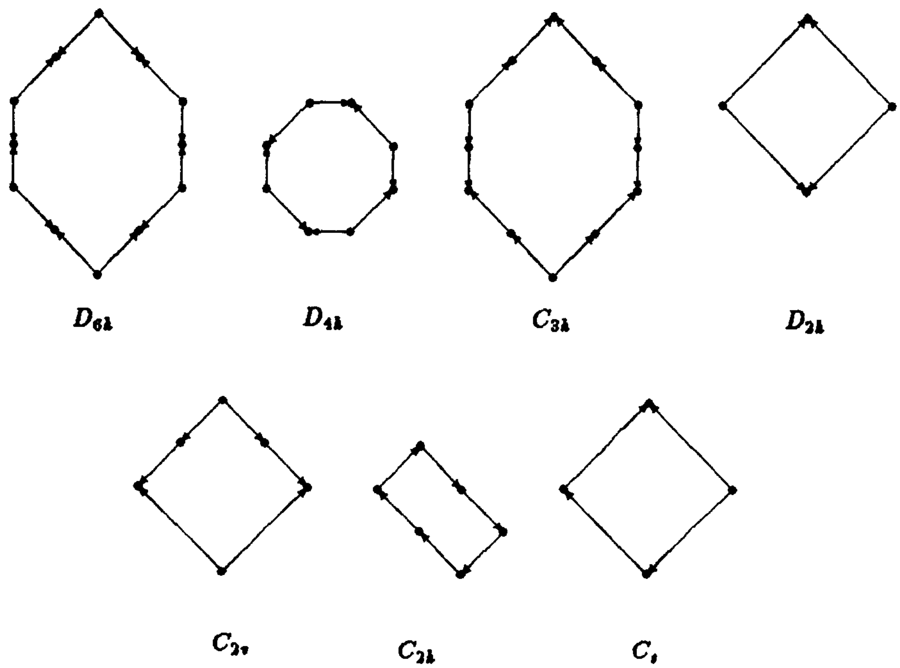

Fig. 2. Some weak cycles and their automorphism groups.

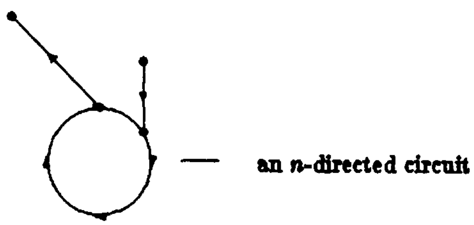

Fig. 3.

Example. Let G be the first weak cycle in Fig. 2. Using the cycle index of D_{6h} we have that the counting series of the near eddy digraphs producd by C is

$$A(x)=\tfrac{1}{12}(\tilde{T}^6(x)+3\tilde{T}^2(x)\tilde{T}^2(x^2)+4\tilde{T}^3(x^2)+2\tilde{T}^2(x^3)+2\tilde{T}(x^6)).$$

To emunerate the digraph of the type (b) in Theorem 3 is a more difficult task. We leave it as an open problem.

Finally we would like to point out that for any n there is a digraph D such that $L(D), L^2(D), \ldots, L^n(D)$ and $L^{n+1}(D)$ are cospectral, as shown in Fig. 3. From this result we can see that there is an arbitrarily large set of cospectral digraphs.

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